Multigluon tree amplitudes with a pair of massive fermions

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Abstract

We consider the calculation of n-point multigluon tree amplitudes with a pair of massive fermions in QCD. We give the explicit transformation rules of this kind of massive fermion-pair amplitudes with respect to different reference momenta and check the correctness of them by SUSY Ward identities. Using these rules and onshell BCFW recursion relation, we calculate the analytic results of several n-point multigluon amplitudes.

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I. INTRODUCTION

Scattering amplitudes are important from both theoretical and experimental points of view. Traditionally we use Feynman diagrams to calculate the scattering amplitudes in field theories. During the past several years, motivated by string theory [1], new efficient methods for tree level amplitudes have been suggested. The offshell CSW rule [2] suggests that the tree amplitudes in gauge theories can be constructed by offshell continued MHV vertices [3]. Then Britto, , Cachazo, Feng and Witten (BCFW) [4–6] gave an onshell recursion relation where the higher point amplitudes can be obtained by lower point onshell amplitudes whose momenta are properly shifted to complex space.

The BCFW recursion relations have been extensively used to calculate tree level in various massless gauge theories [7–14]. Its extension to gravity amplitudes has also been considered [15–23]. Those applications are all related to massless external particles. But in fact, all the matter particles and weak bosons in standard model are massive. Furthermore, massive amplitudes are more important in higher energy physics experiments such as LHC experiments for processes with top quarks, Higgs particles and possible supersymmetric particles. So it is important to consider the amplitudes with massive external particles. In [24–27], the amplitudes with one external massive gauge bosons or Higgs bosons have been discussed. Multigluon amplitudes with pairs of massive scalars or quarks have been studied in [28–36]. An excellent and compact expression for multigluon amplitudes with a pair of massive scalars or quarks and any number of plus helicity gluons has been found in [38] by using an off-shell recursive methods [39] and the BCFW relations. In [40], the authors use some supersymmetric Ward identities to relate a compact expression for multigluon helicity amplitudes involving a pair of massive quarks to amplitudes with massive scalars. A thorough discussion of BCFW onshell recursion relation for amplitudes with massive external particles are given in [32] where the shifted momenta can be massless or massive. The authors also use BCFW recursion relation to obtain compact expressions for multigluon amplitudes involving a pair of massive quarks or scalars and one minus gluon helicity adjacent to fermions. Multigluon amplitudes with a pair of massive scalar and one minus helicity gluon not adjacent to scalars are calculated by a different way [30]. All helicity amplitudes with a pair of massive quarks are calculated in [37] up to six external particles.

In this paper, we use the notation and convention in [32] and explore the calculation of

several n-point helicity amplitudes with a pair of massive quarks and one minus helicity gluon (fermion-pair amplitudes). The massive amplitudes depend on reference momenta which define the helicity of the massive fermions. One can relate the fermion-pair amplitudes with respect to different reference momenta. We first give the explicit form of the transformation rules and check the correctness of them by SUSY Ward identities. Then by shifting the momenta of a massive particle and a gluon and using the BCFW recursion relation, we get the analytic expressions of several n-point fermion-pair amplitudes. In section 2, we review the spinor formalism of massive fermions and obtain the transformation rules for fermion-pair amplitudes defined on different reference momenta. In section 3, we calculate several multigluon amplitudes with a pair of fermions. Section 4 is for the summary.

II. SPINOR FORMALISM OF MASSIVE FERMIONS

We briefly give our notation and convention of spinor helicity formalism. For massless fermions, particles and antiparticles both have definite helicities. Their corresponding spinor states are $u(p,\pm), v(p,\pm)$, which can be denoted as follows[32]:

$$u(p,\pm) = |p\mp\rangle, v(p,\pm) = |p\mp\rangle. \tag{1}$$

For the conjugate states, similar notations are

$$\bar{u}(p,\pm) = \langle\pm p|, \bar{v}(p,\pm) = \langle\pm p|. \tag{2}$$

A massless momentum q^μ can be written in spinor form

$$q^\mu = \frac{1}{2}\langle -q|\gamma^\mu|q-\rangle = \frac{1}{2}\langle q|\gamma^\mu|q] = \frac{1}{2}\langle +q|\gamma^\mu|q+\rangle = \frac{1}{2}[q|\gamma^\mu|q\rangle. \tag{3}$$

The scalar product of two massless momenta p, q can be obtained as

$$2p\cdot q = \langle pq\rangle[qp]. \tag{4}$$

For a massive momentum $K^2 = m^2$, one can always split it into two massless momenta by introducing a reference massless momentum q,

$$K = K^\flat + \frac{K^2}{2K\cdot q}q = K^\flat + \frac{m^2}{2K\cdot q}q, \tag{5}$$

where $(K^\flat)^2 = 0$.

Massive fermions are not helicity eigenstates. Their helicities are frame dependent and we can introduce a null reference momentum q to define their helicity states. In this formalism, the massive fermions and anti-fermions states with momentum $p^2 = m^2$ are

$$
\begin{aligned}
u(p,\pm) &= \frac{1}{\langle p^\flat \mp |q\pm\rangle}(\not{p}+m)|q\pm\rangle, &(6)\\
v(p,\pm) &= \frac{1}{\langle p^\flat \mp |q\pm\rangle}(\not{p}-m)|q\pm\rangle, &(7)\\
\bar{u}(p,\pm) &= \langle q\mp|\frac{1}{\langle p^\flat \mp |q\pm\rangle}(\not{p}+m), &(8)\\
\bar{v}(p,\pm) &= \langle q\mp|\frac{1}{\langle p^\flat \mp |q\pm\rangle}(\not{p}-m). &(9)
\end{aligned}
$$

In the above, $p = p^\flat + \frac{m^2}{2p\cdot q}q$. We can also define massless states $|p^\flat\pm\rangle = \frac{1}{\langle p^\flat \pm |q\mp\rangle}\not{p}|q\mp\rangle$ and rewrite the massive fermion states as

$$
u(p,\pm) = |p^\flat\mp\rangle + \frac{m}{\langle p^\flat \mp |q\pm\rangle}|q\pm\rangle. \tag{10}
$$

We can obtain similar forms for all other states. It is easy to see that we have a smooth massless limit.

Because the massive fermionic helicity states depend on reference momenta, then the amplitudes with massive external fermions should also depend on reference momenta. But we can relate fermionic spinor states with one reference momentum to these with another reference momentum[32]. Let $q, \tilde{q}$ be two light-like reference momenta, we have the following relation between the spinor states corresponding to the two reference momenta,

$$
\begin{pmatrix} \bar{u}(+) \\ \bar{u}(-) \end{pmatrix}_{\tilde{q}} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \bar{u}(+) \\ \bar{u}(-) \end{pmatrix}_{q}, \tag{11}
$$

where

$$
c_{11} = \frac{\langle \tilde{q}|\not{p}|q\rangle}{\langle \tilde{q}\tilde{p}^\flat\rangle[p^\flat q]}, c_{12} = \frac{m\langle \tilde{q}q\rangle}{\langle \tilde{q}\tilde{p}^\flat\rangle\langle p^\flat q\rangle}, c_{21} = \frac{m[\tilde{q}q]}{[\tilde{q}\tilde{p}^\flat][p^\flat q]}, c_{22} = \frac{[\tilde{q}|\not{p}|q\rangle}{[\tilde{q}\tilde{p}^\flat]\langle p^\flat q\rangle}. \tag{12}
$$

In the above equations, $\tilde{p}^\flat(p^\flat)$ means splitting p with respect to $\tilde{q}(q)$. Similar relations can be obtained for other massive fermionic states.

In this paper, we consider n-point amplitudes with gluons and a pair of massive fermions $A(1_Q, 2, 3, \cdots, n-1, n_{\bar{Q}})$. In the calculation, we choose the same reference momentum for the external massive fermions. An amplitude with reference momentum q for the external fermions is denoted by $A_q(1_Q, 2, 3, \cdots, n-1, n_{\bar{Q}})$. In contrast to amplitudes with

massless fermions, there are both helicity-conserving and helicity-flipping amplitudes for massive amplitudes. With the same external gluons, there are four different helicity amplitudes $A_q(1^+_Q, 2, 3, \cdots, n-1, n^+_{\bar{Q}})$, $A_q(1^+_Q, 2, 3, \cdots, n-1, n^-_{\bar{Q}})$, $A_q(1^-_Q, 2, 3, \cdots, n-1, n^+_{\bar{Q}})$, $A_q(1^-_Q, 2, 3, \cdots, n-1, n^-_{\bar{Q}})$. Similar to the relation (11), we can get relations between amplitudes with respect to different reference momenta,

$$\begin{pmatrix} A^{++}_{\tilde{q}} \\ A^{+-}_{\tilde{q}} \\ A^{-+}_{\tilde{q}} \\ A^{--}_{\tilde{q}} \end{pmatrix} = \begin{pmatrix} C^1_{11}C^n_{11} & C^1_{11}C^n_{12} & C^1_{12}C^n_{11} & C^1_{12}C^n_{12} \\ C^1_{11}C^n_{21} & C^1_{11}C^n_{22} & C^1_{12}C^n_{21} & C^1_{12}C^n_{22} \\ C^1_{21}C^n_{11} & C^1_{21}C^n_{12} & C^1_{22}C^n_{11} & C^1_{22}C^n_{12} \\ C^1_{21}C^n_{21} & C^1_{21}C^n_{22} & C^1_{22}C^n_{21} & C^1_{22}C^n_{22} \end{pmatrix} \begin{pmatrix} A^{++}_q \\ A^{+-}_q \\ A^{-+}_q \\ A^{--}_q \end{pmatrix}. \tag{13}$$

In the above, A^{++}_q is shorthand notation of $A_q(1^+_Q, \cdots, n^+_{\bar{Q}})$ and all the amplitudes have the same external gluons. As in equation (12), the elements of the matrix are

$$\begin{aligned} C^1_{11} &= \frac{\langle\tilde{q}|\not{p}_1|q]}{\langle\tilde{q}\tilde{p_1}^\flat\rangle[p^\flat_1 q]}, C^1_{12} = \frac{m\langle\tilde{q}q\rangle}{\langle\tilde{q}\tilde{p_1}^\flat\rangle\langle p^\flat_1 q\rangle}, C^1_{21} = \frac{m[\tilde{q}q]}{[\tilde{q}\tilde{p_1}^\flat][p^\flat_1 q]}, C^1_{22} = \frac{[\tilde{q}|\not{p}_1|q\rangle}{[\tilde{q}\tilde{p_1}^\flat]\langle p^\flat_1 q\rangle}, \\ C^n_{11} &= \frac{\langle\tilde{q}|\not{p}_n|q]}{\langle\tilde{q}\tilde{p_n}^\flat\rangle[p^\flat_n q]}, C^n_{12} = \frac{m\langle\tilde{q}q\rangle}{\langle\tilde{q}\tilde{p_n}^\flat\rangle\langle p^\flat_n q\rangle}, C^n_{21} = \frac{m[\tilde{q}q]}{[\tilde{q}\tilde{p_n}^\flat][p^\flat_n q]}, C^n_{22} = \frac{[\tilde{q}|\not{p}_n|q\rangle}{[\tilde{q}\tilde{p_n}^\flat]\langle p^\flat_n q\rangle}. \end{aligned} \tag{14}$$

These relations between massive amplitudes are important for our calculations in the next section and the correctness of it will also be checked there.

III. CALCULATION OF FERMION-PAIR AMPLITUDES

In this section, we use the onshell BCFW recursion relations to calculate several n-point amplitudes with a pair of massive fermions and $n-2$ gluons. First we list several excellent and useful results about amplitudes with a pair of massive particles, which are the building blocks of our calculation.

An excellent and compact expression for the amplitude of a massive complex scalar-antiscalar pair and any number of positive helicity gluons is obtained in [38]:

$$A(1_\phi, 2^+, \cdots, (n-1)^+, n_{\bar{\phi}}) = 2^{n/2-1} i m^2 \frac{[2|\prod_{k=3}^{n-2}(y_{1,k} - \not{p}_k \not{p}_{1,k-1})|n-1]}{y_{1,2}y_{1,3}\cdots y_{1,n-2}\langle 23\rangle\langle 34\rangle\cdots\langle n-2, n-1\rangle}, \tag{15}$$

where

$$p_{1,k} = p_1 + p_2 + \cdots + p_k, \tag{16}$$

$$y_{1,k} = (p_1 + p_2 + \cdots + p_k)^2 - m^2. \tag{17}$$

In supersymmetric theories, SUSY transformation can relate a bosonic particle state to a fermionic particle state. And there are also relations between amplitudes related by SUSY transformations. The SUSY transformations of helicity states have been discussed in [41, 42]. In massless SUSY QCD theory, these SUSY transformations have been applied to the helicity amplitudes [43–46]. In massive SUSY QCD, similar transformations of helicity states have been derived in [40]. Using SUSY transformations, some relations between amplitudes with different external helicity particles can also be obtained. These relations are the so-called SUSY Ward identities[40]. As already mentioned, in this paper, we choose the same reference momentum for the external massive fermions of an amplitude. In this case, amplitudes of a massive fermion pair and all plus helicity gluons have the following relations with amplitudes where the fermion pair is replaced by massive scalar pair,

$$
\begin{aligned}
A_q(1_Q^+, 2^+, \cdots, (n-1)^+, n_{\bar{Q}}^-) &= \frac{\langle p_n^\flat q\rangle}{\langle p_1^\flat q\rangle} A(1_\phi^+, 2^+, \cdots, (n-1)^+, n_{\bar{\phi}}^-), \\
A_q(1_Q^-, 2^+, \cdots, (n-1)^+, n_{\bar{Q}}^+) &= -\frac{\langle p_1^\flat q\rangle}{\langle p_n^\flat q\rangle} A(1_\phi^-, 2^+, \cdots, (n-1)^+, n_{\bar{\phi}}^+), \\
A_q(1_Q^-, 2^+, \cdots, (n-1)^+, n_{\bar{Q}}^-) &= \frac{\langle p_1^\flat p_n^\flat\rangle}{m} A(1_\phi^-, 2^+, \cdots, (n-1)^+, n_{\bar{\phi}}^-).
\end{aligned} \tag{18}
$$

It is noted that all the SUSY Ward identities as the above are derived by using supersymmetry and these relations should hold to any perturbative orders. At the tree level some relations can be applied to non-supersymmetric theory. This results from the fact that at tree level the quark-gluon amplitudes are the same in both SUSY and non-SUSY theories and at loop level there are contributions from SUSY particles to amplitudes. The helicity flipping amplitude with all plus external particles vanish, $A_q(1_Q^+, 2^+, \cdots, (n-1)^+, n_{\bar{Q}}^+) = 0$. Plugging eq.(15) into eq.(18), we can get the basic building blocks for the calculation of massive fermion-pair amplitudes.

There are also similar relations between fermion-pair amplitudes and scalar-pair amplitudes with one minus helicity gluon

$$
\begin{aligned}
A_j(1_Q^+, 2^+, \cdots, j^-, \cdots, (n-1)^+, n_{\bar{Q}}^-) &= \frac{\langle n_j j\rangle}{\langle 1_j j\rangle} A(1_\phi^+, 2^+, \cdots, j^-, \cdots, (n-1)^+, n_{\bar{\phi}}^-), \\
A_j(1_Q^-, 2^+, \cdots, j^-, \cdots, (n-1)^+, n_{\bar{Q}}^+) &= -\frac{\langle 1_j j\rangle}{\langle n_j j\rangle} A(1_\phi^-, 2^+, \cdots, j^-, \cdots, (n-1)^+, n_{\bar{\phi}}^+),
\end{aligned} \tag{19}
$$

where $1_j = p_1^\flat|_{r=p_j}$ is the projection of p_1 when we choose p_j as reference momentum. One can note that there is not similar relations for helicity flipping amplitudes when there are one minus helicity gluon.

Before we precede to the calculation, here we use the SUSY Ward identities (18) to check the correctness of the transformation rules (13). We know that $A_q(1_Q^+, 2^+, \cdots, (n-1)^+, n_{\bar{Q}}^+) = 0$ is correct for any reference momentum q. Then using the transformation rules (13), we can obtain

$$
\begin{aligned}
A_{\tilde{q}}(1_Q^+, 2^+, \cdots, (n-1)^+, n_{\bar{Q}}^+) &= C_{11}^1 C_{11}^n A_q^{++} + C_{11}^1 C_{12}^n A_q^{+-} + C_{12}^1 C_{11}^n A_q^{-+} + C_{12}^1 C_{12}^n A_q^{--} \\
&= C_{11}^1 C_{12}^n A_q^{+-} + C_{12}^1 C_{11}^n A_q^{-+} + C_{12}^1 C_{12}^n A_q^{--} \\
&= (C_{11}^1 C_{12}^n \frac{\langle nq \rangle}{\langle 1q \rangle} - C_{12}^1 C_{11}^n \frac{\langle 1q \rangle}{\langle nq \rangle} + C_{12}^1 C_{12}^n \frac{\langle 1n \rangle}{m}) A(\phi), \qquad (20)
\end{aligned}
$$

where $A(\phi) = A(1_\phi^+, 2^+, \cdots, (n-1)^+, n_{\bar{\phi}}^-)$. The coefficient of $A(\phi)$ can be calculated as

$$
\begin{aligned}
&\frac{\langle \tilde{q}|\not{p}_1|q]}{\left\langle \tilde{q}\tilde{1} \right\rangle [1q]} \frac{m \langle \tilde{q}q \rangle}{\langle \tilde{q}\tilde{n} \rangle \langle nq \rangle} \frac{\langle nq \rangle}{\langle 1q \rangle} - \frac{\langle \tilde{q}|\not{p}_n|q]}{\langle \tilde{q}\tilde{n} \rangle [nq]} \frac{m \langle \tilde{q}q \rangle}{\left\langle \tilde{q}\tilde{1} \right\rangle \langle 1q \rangle} \frac{\langle 1q \rangle}{\langle nq \rangle} + \frac{m \langle \tilde{q}q \rangle}{\left\langle \tilde{q}\tilde{1} \right\rangle \langle 1q \rangle} \frac{m \langle \tilde{q}q \rangle}{\langle \tilde{q}\tilde{n} \rangle \langle nq \rangle} \frac{\langle 1n \rangle}{m} \\
&= \frac{\langle \tilde{q}1 \rangle}{\left\langle \tilde{q}\tilde{1} \right\rangle} \frac{m \langle \tilde{q}q \rangle}{\langle \tilde{q}\tilde{n} \rangle \langle nq \rangle} \frac{\langle nq \rangle}{\langle 1q \rangle} - \frac{\langle \tilde{q}n \rangle}{\langle \tilde{q}\tilde{n} \rangle} \frac{m \langle \tilde{q}q \rangle}{\left\langle \tilde{q}\tilde{1} \right\rangle \langle 1q \rangle} \frac{\langle 1q \rangle}{\langle nq \rangle} + \frac{\langle 1n \rangle \langle \tilde{q}q \rangle}{\left\langle \tilde{q}\tilde{1} \right\rangle \langle 1q \rangle} \frac{m \langle \tilde{q}q \rangle}{\langle \tilde{q}\tilde{n} \rangle \langle nq \rangle} \\
&= \frac{m \langle \tilde{q}q \rangle}{\left\langle \tilde{q}\tilde{1} \right\rangle \langle \tilde{q}\tilde{n} \rangle \langle nq \rangle \langle 1q \rangle} (\langle \tilde{q}1 \rangle \langle nq \rangle - \langle \tilde{q}n \rangle \langle 1q \rangle + \langle 1n \rangle \langle \tilde{q}q \rangle) \\
&= 0. \qquad (21)
\end{aligned}
$$

So from the transformation rules, we can obtain the correct results $A_{\tilde{q}}(\phi) = 0$. In a similar way, one can check the correctness of the transformation rules (13) from other SUSY Ward identities.

From the SUSY Ward identities, we have obtained the fermion-pair amplitudes with all plus helicity gluons. Then we can use onshell BCFW recursion relation to get fermion-pair amplitudes with other gluon helicity configurations. In [32], it has been proved that by choosing a proper momenta shift, we can always use onshell BCFW recursion relation to calculate fermion-pair amplitudes. In the following, we will calculate several concrete multigluon amplitudes.

The amplitude $A(1_Q^+, 2^+, \cdots, (n-1)^-, n_{\bar{Q}}^-)$ can be calculated by shifting momenta p_{n-1}, p_n. In spinor formalism, it is

$$
\begin{aligned}
|n\rangle &\to |n\rangle + z|n-1\rangle, \\
|n-1] &\to |n-1] - z|n], \qquad (22)
\end{aligned}
$$

where

$$
|n\pm\rangle = |p_n^\flat \pm\rangle, \qquad (23)
$$

and

$$p_n = p_n^\flat + \frac{m^2}{2p_{n-1}\cdot p_n} p_{n-1}. \tag{24}$$

Then using BCFW recursion relation, we can get a compact result as follows,

$$\begin{aligned}
& A_{n-1}(1_Q^+, 2^+, \cdots, (n-1)^-, n_{\bar{Q}}^-) \\
= & \sum_{k=2}^{n-2} A_{n-1}(1_Q^+, \cdots, (k-1)^+, \hat{P}_{k,n-1}^+, \hat{n}_{\bar{Q}}^-) \frac{i}{p_{k,n-1}^2} A(k^+, \cdots, \widehat{n-1}^-, -\hat{P}_{k,n-1}^-) \\
= & i2^{n/2-1} \frac{\langle n, n-1\rangle}{\langle 1, n-1\rangle \langle 23\rangle \cdots \langle n-2, n-1\rangle} \sum_{k=2}^{n-2} \frac{\langle n-1|\not{p}_{k,n-1}\not{p}_n|n-1\rangle^2}{p_{k,n-1}^2 \langle k|\not{p}_{k,n-1}\not{p}_n|n-1\rangle} \times \\
& \left(\delta_{k,2} + \delta_{k\neq 2} \frac{m^2 \langle k-1, k\rangle [2| \prod_{j=3}^{k-1}(y_{1,j} - \not{p}_j \not{p}_{1,j-1})\not{p}_{k,n-1}|n-1\rangle}{y_{1,2}\cdots y_{1,k-1}\langle k-1|\not{p}_{k,n-1}\not{p}_n|n-1\rangle}\right),
\end{aligned} \tag{25}$$

where $\delta_{k\neq 2} = 1 - \delta_{k,2}$, and when $k = 3$, $\prod_{j=3}^{k-1}(\cdots) = 1$. From the SUSY Ward identities, if we miss the factor $\frac{\langle n,n-1\rangle}{\langle 1,n-1\rangle}$, we get the corresponding multigluon amplitude with massive scalar. It has a more compact form than the one obtained in [30] because we use more compact amplitudes eq.(15) as building blocks. But we can check some lower point amplitudes with others. The four point scalar amplitude is

$$A(1_\phi^+, 2^+, 3^-, 4_\phi^-) = i2 \frac{\langle 3|\not{p}_{2,3}\not{p}_4|3\rangle^2}{\langle 23\rangle\, p_{2,3}^2 \langle 2|\not{p}_{2,3}\not{p}_4|3\rangle} = i2 \frac{\langle 3|\not{p}_4|2]}{p_{2,3}^2 y_{1,2}}. \tag{26}$$

The five point amplitude is

$$\begin{aligned}
A(1_\phi^+, 2^+, 3^+, 4^-, 5_\phi^-) \;=\; & i2^{3/2} \frac{\langle 4|\not{p}_{2,4}\not{p}_5|4\rangle^2}{\langle 23\rangle \langle 34\rangle\, p_{2,4}^2 \langle 2|\not{p}_{2,4}\not{p}_5|4\rangle} \\
& + i2^{3/2} \frac{m^2 \langle 4|\not{p}_{3,4}\not{p}_5|4\rangle^2}{\langle 34\rangle\, p_{3,4}^2 \langle 3|\not{p}_{3,4}\not{p}_5|4\rangle} \frac{[2|\not{p}_{3,4}|4\rangle}{y_{1,2}\langle 2|\not{p}_{3,4}\not{p}_5|4\rangle} \\
= \; & i2^{3/2} \frac{\langle 4|\not{p}_1\not{p}_{2,4}|4\rangle^2}{\langle 23\rangle \langle 34\rangle\, p_{2,4}^2 \langle 2|\not{p}_1\not{p}_{2,4}|4\rangle} + i2^{3/2} \frac{m^2 [3|\not{p}_5|4\rangle^2 [23]}{[34] y_{1,2} y_{1,3} \langle 4|\not{p}_5\not{p}_{3,4}|2\rangle}
\end{aligned} \tag{27}$$

These results are the same as the ones obtained from other ways [28, 30] up to overall conventional coefficients.

Then we calculate another four multigluon amplitudes with a massive fermion-pair. For amplitude

$$A(1_Q^+, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^+),$$

we shift the momenta p_1 and p_3,

$$\begin{aligned}
|1\rangle &\to |1\rangle + z|3\rangle, \\
|3] &\to |3] - z|1].
\end{aligned} \tag{28}$$

The amplitude $A(1_Q^+, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^+)$ can be decomposed as

$$\begin{aligned} A_3(1_Q^+, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^+) &= \sum_{k=4}^{n-1} A_3(\hat{1}_Q^+, 2^+, \hat{P}_{3,k}^+, \cdots, n_{\bar{Q}}^+) \frac{i}{p_{3,k}^2} A(-\hat{P}_{3,k}^-, \hat{3}^-, \cdots, k^+) \\ &+ \sum_{l=3}^{n-1} A_3(\hat{1}_Q^+, 2^+, \hat{P}_{2,l}^+, \cdots, n_{\bar{Q}}^+) \frac{i}{p_{2,l}^2} A(-\hat{P}_{2,l}^-, 2^+, \hat{3}^-, \cdots, l^+) \\ &+ A_3(\hat{1}_Q^+, 2^+, -\hat{P}_{12}^-) \frac{1}{p_{12}^2} A_3(\hat{P}_{12}^+, \hat{3}^-, \cdots, n_{\bar{Q}}^+). \end{aligned} \tag{29}$$

It is easy to see the first two terms in the above equation are both zero because there are fermion-pair amplitudes with all plus helicity. Let us see the third term $A_3(\hat{1}_Q^+, 2^+, -\hat{P}_{12}^-) \frac{1}{p_{12}^2} A_3(\hat{P}_{12}^+, \hat{3}^-, \cdots, n_{\bar{Q}}^+)$. We already know

$$A_{\hat{3}}(\hat{P}_{12}^+, \hat{3}^-, \cdots, n_{\bar{Q}}^+) = 0. \tag{30}$$

Using the transformation of amplitude with respect to different momenta, it is easy to show

$$A_3(\hat{P}_{12}^+, \hat{3}^-, \cdots, n_{\bar{Q}}^+) = 0. \tag{31}$$

Then

$$A_3(1_Q^+, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^+) = 0. \tag{32}$$

Using the same recursive calculation and induction, we can prove

$$A_j(1_Q^+, 2^+, \cdots, j^-, \cdots, (n-1)^+, n_{\bar{Q}}^+) = 0. \tag{33}$$

This is consistent with the result from SUSY Ward identities[40], which are obtained only by the supersymmetry of massive SUSY QCD.

Then we use the same shifting of momenta as eq.(28) and calculate the amplitude $A_3(1_Q^+, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^-)$. We obtain

$$\begin{aligned} & A_3(1_Q^+, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^-) \\ &= i 2^{n/2-1} \frac{\langle n3 \rangle}{\langle 13 \rangle} \frac{m^2}{\langle 34 \rangle \cdots \langle n-2, n-1 \rangle} \\ &\times \sum_{k=4}^{n-1} \frac{\langle 3|\not{p}_1 \not{p}_{3,k}|3\rangle^3}{p_{3,k}^2 \langle 3|\not{p}_1 \not{p}_{3,k}|2\rangle \langle 3|\not{p}_1 \not{p}_{3,k}|k\rangle (\langle 3|\not{p}_1 \not{p}_{3,k}|3\rangle y_{1,2} + \langle 3|\not{p}_1 \not{p}_{1,2}|3\rangle p_{3,k}^2)} \\ &\times \{ \frac{\delta_{k \neq n-1} \langle k, k+1 \rangle \langle 3|\not{p}_1 \not{p}_{3,k}|3\rangle}{y_{1,k} \cdots y_{1,n-2} \langle 3|\not{p}_1 \not{p}_{3,k}|k+1\rangle} ([2|(y_{1,k} + \not{p}_{3,k} \not{p}_{k+1,n}) \prod_{j=k+1}^{n-2} (y_{1,j} - \not{p}_j \not{p}_{1,j-1})|n-1] \end{aligned} \tag{34}$$

$$
\begin{aligned}
+ & \frac{\langle 3|\not{p}_1|2]}{\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle} p_{3,k}^2 \langle 3|\not{p}_{k+1,n} \prod_{j=k+1}^{n-2} (y_{1,j} - \not{p}_j \not{p}_{1,j-1})|n-1]) + \delta_{k,n-1}\langle 3|\not{p}_{3,n-1}|2]\} \\
+ & i2^{n/2-1}\frac{\langle n3\rangle}{\langle 13\rangle}\frac{1}{\langle 23\rangle\cdots\langle n-2,n-1\rangle}\sum_{l=3}^{n-1}\frac{\langle 3|\not{p}_1\not{p}_{2,l}|3\rangle^3}{p_{2,l}^2\langle 3|\not{p}_1\not{p}_{2,l}|2\rangle\langle 3|\not{p}_1\not{p}_{2,l}|l\rangle} \\
\times & \{\delta_{l,n-1} + \delta_{l\neq n-1} m^2 \frac{\langle l,l+1\rangle\langle 3|\not{p}_{2,l}\prod_{j=l+1}^{n-2}(y_{1,j} - \not{p}_j\not{p}_{1,j-1})|n-1]}{y_{1,l}\cdots y_{1,n-2}\langle 3|\not{p}_1\not{p}_{2,l}|l+1\rangle}\} \\
+ & i2^{n/2-1}\frac{\langle n3\rangle}{\langle 13\rangle}\frac{\langle 3|\not{p}_1|2]}{y_{1,2}\langle 23\rangle\cdots\langle n-2,n-1\rangle} \\
\times & \sum_{j=4}^{n-1}\frac{\langle 3|\not{p}_1\not{p}_2|3\rangle\langle 3|\not{p}_{1,2}\not{p}_{3,j}|3\rangle^2}{(\langle 3|\not{p}_1\not{p}_2|3\rangle p_{3,j}^2 + \langle 3|\not{p}_1\not{p}_{3,j}|3\rangle y_{1,2})\langle 3|(y_{1,2}+\not{p}_{1,2}\not{p}_{3,j})|j\rangle} \\
\times & \{\delta_{j,n-1} + \delta_{j\neq n-1} m^2 \frac{\langle j,j+1\rangle\langle 3|\not{p}_{3,j}\prod_{k=j+1}^{n-2}(y_{1,k} - \not{p}_k\not{p}_{1,k-1})|n-1]}{y_{1,j}\cdots y_{1,n-2}\langle 3|(y_{1,2}+\not{p}_{1,2}\not{p}_{3,j})|j+1\rangle}\}.
\end{aligned}
$$

For multigluon amplitude $A_3(1_Q^-, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^+)$, we can use the similar recursive method to calculate and we obtain

$$
\begin{aligned}
& A_3(1_Q^-, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^+) \qquad (35) \\
= & -i2^{n/2-1}\frac{\langle 13\rangle}{\langle n3\rangle}\frac{m^2}{\langle 34\rangle\cdots\langle n-2,n-1\rangle} \\
\times & \sum_{k=4}^{n-1}\frac{\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle^3}{p_{3,k}^2\langle 3|\not{p}_1\not{p}_{3,k}|2\rangle\langle 3|\not{p}_1\not{p}_{3,k}|k\rangle(\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle y_{1,2} + \langle 3|\not{p}_1\not{p}_{1,2}|3\rangle p_{3,k}^2)} \\
\times & \{\frac{\delta_{k\neq n-1}\,\langle k,k+1\rangle\,\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle}{y_{1,k}\cdots y_{1,n-2}\langle 3|\not{p}_1\not{p}_{3,k}|k+1\rangle}([2|(y_{1,k}+\not{p}_{3,k}\not{p}_{k+1,n})\prod_{j=k+1}^{n-2}(y_{1,j} - \not{p}_j\not{p}_{1,j-1})|n-1] \\
+ & \frac{\langle 3|\not{p}_1|2]}{\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle} p_{3,k}^2 \langle 3|\not{p}_{k+1,n} \prod_{j=k+1}^{n-2} (y_{1,j} - \not{p}_j \not{p}_{1,j-1})|n-1]) + \delta_{k,n-1}\langle 3|\not{p}_{3,n-1}|2]\} \\
- & i2^{n/2-1}\frac{\langle 13\rangle}{\langle n3\rangle}\frac{1}{\langle 23\rangle\cdots\langle n-2,n-1\rangle}\sum_{l=3}^{n-1}\frac{\langle 3|\not{p}_1\not{p}_{2,l}|3\rangle^3}{p_{2,l}^2\langle 3|\not{p}_1\not{p}_{2,l}|2\rangle\langle 3|\not{p}_1\not{p}_{2,l}|l\rangle} \\
\times & \{\delta_{l,n-1} + \delta_{l\neq n-1} m^2 \frac{\langle l,l+1\rangle\langle 3|\not{p}_{2,l}\prod_{j=l+1}^{n-2}(y_{1,j} - \not{p}_j\not{p}_{1,j-1})|n-1]}{y_{1,l}\cdots y_{1,n-2}\langle 3|\not{p}_1\not{p}_{2,l}|l+1\rangle}\} \\
- & i2^{n/2-1}\frac{\langle 13\rangle}{\langle n3\rangle}\frac{\langle 3|\not{p}_1|2]}{y_{1,2}\langle 23\rangle\cdots\langle n-2,n-1\rangle} \\
\times & \sum_{j=4}^{n-1}\frac{\langle 3|\not{p}_1\not{p}_2|3\rangle\langle 3|\not{p}_{1,2}\not{p}_{3,j}|3\rangle^2}{(\langle 3|\not{p}_1\not{p}_2|3\rangle p_{3,j}^2 + \langle 3|\not{p}_1\not{p}_{3,j}|3\rangle y_{1,2})\langle 3|(y_{1,2}+\not{p}_{1,2}\not{p}_{3,j})|j\rangle} \\
\times & \{\delta_{j,n-1} + \delta_{j\neq n-1} m^2 \frac{\langle j,j+1\rangle\langle 3|\not{p}_{3,j}\prod_{k=j+1}^{n-2}(y_{1,k} - \not{p}_k\not{p}_{1,k-1})|n-1]}{y_{1,j}\cdots y_{1,n-2}\langle 3|(y_{1,2}+\not{p}_{1,2}\not{p}_{3,j})|j+1\rangle}\}.
\end{aligned}
$$

Comparing this result with eq.(34), we can see that they are different from each other just by a constant coefficient. And this is consistent with the SUSY Ward identities in eq.(19). Just as eq.(25), eq.(34) and eq.(35) have more compact forms than the ones obtained from other ways.

Then let us check the massless limit of eq.(34). Taking $m = 0$, eq.(34) becomes

$$\begin{aligned}
& A_3(1_Q^+, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^-) \qquad (36) \\
= \; & i2^{n/2-1}\frac{\langle n3\rangle}{\langle 13\rangle}\frac{1}{\langle 23\rangle\cdots\langle n-2,n-1\rangle}\frac{\langle 3|\not{p}_{n-1}\not{p}_{2,n-1}|3\rangle^3}{p_{2,n-1}^2\langle 3|\not{p}_{n-1}\not{p}_{2,n-1}|2\rangle\langle 3|\not{p}_{n-1}\not{p}_{2,n-1}|n-1\rangle} \\
+ \; & i2^{n/2-1}\frac{\langle n3\rangle}{\langle 13\rangle}\frac{\langle 3|\not{p}_1|2]}{y_{1,2}\langle 23\rangle\cdots\langle n-2,n-1\rangle} \\
\times \; & \frac{\langle 3|\not{p}_1\not{p}_2|3\rangle\langle 3|\not{p}_{1,2}\not{p}_{3,n-1}|3\rangle^2}{(\langle 3|\not{p}_1\not{p}_2|3\rangle p_{3,n-1}^2 + \langle 3|\not{p}_1\not{p}_{3,n-1}|3\rangle y_{1,2})\langle 3|(y_{1,2}+\not{p}_{1,2}\not{p}_{3,n-1})|n-1\rangle} \\
= \; & i2^{n/2-1}\frac{\langle n3\rangle^3\langle 31\rangle}{\langle 12\rangle\cdots\langle n1\rangle}.
\end{aligned}$$

This is exactly the MHV amplitude with a fermion-antifermion pair in massless QCD theory.

Finally, we calculate the amplitude $A_3(1_Q^-, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^-)$. There is no SUSY Ward identity to relate this amplitude with the corresponding scalar one. So this kind of amplitudes should be calculated directly from lower point amplitudes with a pair of fermions using onshell recursive method. The result is

$$\begin{aligned}
& A_3(1_Q^-, 2^+, 3^-, \cdots, (n-1)^+, n_{\bar{Q}}^-) \qquad (37) \\
= \; & \frac{i2^{n/2-1}m}{\langle 34\rangle\cdots\langle n-2,n-1\rangle} \\
\times \; & \sum_{k=4}^{n-1}(\langle 1n\rangle + \frac{\langle 3n\rangle}{\langle 3|\not{p}_{3,k}|1]}p_{3,k}^2)\frac{\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle^3}{p_{3,k}^2\langle 3|\not{p}_1\not{p}_{3,k}|2\rangle\langle 3|\not{p}_1\not{p}_{3,k}|k\rangle(\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle y_{1,2} + \langle 3|\not{p}_1\not{p}_{1,2}|3\rangle p_{3,k}^2)} \\
\times \; & \{\frac{\delta_{k\neq n-1}\,\langle k,k+1\rangle\,\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle}{y_{1,k}\cdots y_{1,n-2}\langle 3|\not{p}_1\not{p}_{3,k}|k+1\rangle}([2|(y_{1,k}+\not{p}_{3,k}\not{p}_{k+1,n})\prod_{j=k+1}^{n-2}(y_{1,j}-\not{p}_j\not{p}_{1,j-1})|n-1] \\
+ \; & \frac{\langle 3|\not{p}_1|2]}{\langle 3|\not{p}_1\not{p}_{3,k}|3\rangle}p_{3,k}^2\langle 3|\not{p}_{k+1,n}\prod_{j=k+1}^{n-2}(y_{1,j}-\not{p}_j\not{p}_{1,j-1})|n-1]) + \delta_{k,n-1}\langle 3|\not{p}_{3,n-1}|2]\} \\
+ \; & \frac{i2^{n/2-1}}{m\langle 23\rangle\cdots\langle n-2,n-1\rangle}\sum_{l=3}^{n-1}(\langle 1n\rangle + \frac{\langle 3n\rangle}{\langle 3|\not{p}_{2,l}|1]}p_{2,l}^2)\frac{\langle 3|\not{p}_1\not{p}_{2,l}|3\rangle^3}{p_{2,l}^2\langle 3|\not{p}_1\not{p}_{2,l}|2\rangle\langle 3|\not{p}_1\not{p}_{2,l}|l\rangle} \\
\times \; & \{\delta_{l,n-1} + \delta_{l\neq n-1}m^2\frac{\langle l,l+1\rangle\langle 3|\not{p}_{2,l}\prod_{j=l+1}^{n-2}(y_{1,j}-\not{p}_j\not{p}_{1,j-1})|n-1]}{y_{1,l}\cdots y_{1,n-2}\langle 3|\not{p}_1\not{p}_{2,l}|l+1\rangle}\}
\end{aligned}$$

$$+ \frac{i2^{n/2-1}}{\langle 34\rangle\cdots\langle n-2,n-1\rangle}\sum_{j=4}^{n-1}\frac{\langle 3|\not{p}_1\not{p}_2|3\rangle\langle 3|\not{p}_{1,2}\not{p}_{3,j}|3\rangle^2 C(j)}{(\langle 3|\not{p}_1\not{p}_2|3\rangle p_{3,j}^2+\langle 3|\not{p}_1\not{p}_{3,j}|3\rangle y_{1,2})\langle 3|(y_{1,2}+\not{p}_{1,2}\not{p}_{3,j})|j\rangle}$$
$$\times \ \{\delta_{j,n-1}+\delta_{j\neq n-1}m^2\frac{\langle j,j+1\rangle\langle 3|\not{p}_{3,j}\prod_{k=j+1}^{n-2}(y_{1,k}-\not{p}_k\not{p}_{1,k-1})|n-1]}{y_{1,j}\cdots y_{1,n-2}\langle 3|(y_{1,2}+\not{p}_{1,2}\not{p}_{3,j})|j+1\rangle}\},$$

where $C(j)$ is

$$\begin{aligned} C(j) \ &= \ \frac{m[23]\langle n|\not{p}_3|2]}{y_{1,2}[31]\langle 3|\not{p}_{1,2}|3]}-\frac{m\langle 13\rangle^2\langle n|\not{p}_3|1]}{y_{1,3}\langle 23\rangle^2\langle 3|\not{p}_{1,2}|3]}-\frac{m\langle 13\rangle\langle 3|\not{p}_1|3]}{\langle 23\rangle\langle 3|\not{p}_n|3]\langle 2|(\not{p}_1^\flat+\not{p}_3)|n]} \\ &+ \ \frac{\langle 13\rangle\langle 3|\not{p}_1|2]}{m\langle 23\rangle y_{1,2}}(\frac{y_{1,3}\langle n|\not{p}_{1,2}|1]-m^2\langle n|\not{p}_3|1]}{y_{1,3}\langle 3|\not{p}_{1,2}|1]}-\frac{y_{1,2}\langle n|\not{p}_n|1]}{y_{1,2}\langle 3|\not{p}_n|1]+2p_3\cdot p_n\langle 3|\not{p}_2|1]}) \\ &+ \ \frac{\langle 13\rangle\langle n3\rangle\langle 3|\not{p}_1|2]}{m\langle 23\rangle y_{1,2}\langle 3|(y_{1,2}+\not{p}_{1,2}\not{p}_{3,j})|3\rangle}(p_{3,j}^2+y_{1,2}\frac{\langle 3|\not{p}_{3,j}|1]}{\langle 3|\not{p}_2|1]}). \end{aligned} \tag{38}$$

In principle, for an n-point massive fermion-pair amplitude with definite minus helicity gluon j , we can use the above mentioned shifting to get the analytic expression for it from amplitudes with one minus helicity gluon nearer to the fermions. In fact, it is difficult for doing it by hand except for some special gluon helicity configurations. But because the shift of momenta, recursion decomposition and transformation of amplitudes with respect to different reference momenta are all systematic procedures, so it is suitable to develop a program to do the work.

IV. SUMMARY

In this paper, using onshell BCFW recursion relation and shifting the momenta of a massive fermion and a gluon, we calculate several tree level n-point amplitudes with a massive fermion-antifermion pair and one minus helicity gluon. Amplitudes with massive fermions depend on reference momentum for defining the helicity states of massive external fermions, so it is more difficult for calculating them. We give the explicit transformation rules for amplitudes with respect to different reference momenta, which are important in the analytic calculation of massive fermion-pair amplitudes. The correctness of these rules are checked by SUSY Ward identities. We use the most compact and excellent results for some special gluon helicity configurations, such as all plus helicity, and pure gluon amplitudes as building blocks to obtain four n-point massive amplitudes with more complicated gluon helicity

configurations. Generally, calculating the analytic results of n-point massive amplitudes by hand is difficult. A program need to be developed to calculate the amplitudes. But it is still interesting and calculable to use the recursion method to explore the analytic results of amplitudes with finite external particles and more fermions, which are important in high energy physics experiments.

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